Exercise 1

a $Ax = b$ $\quad A = \underbrace{D}_{\text{diagonal of } A} - B \;\to\; Dx = b + Bx$

$$\to \quad Dx^{(n+1)} = b + Bx^{(n)}$$
$$Dx = b + Bx$$
$$D(x - x^{(n+1)}) = B(x - x^{(n)}) \;\to\; x - x^{(n+1)} = \underbrace{D^{-1}B}_{\text{iteration matrix}}(x - x^{(n)})$$

$$\rho(D^{-1}B) \le \|D^{-1}B\|_\infty = \max_i \sum_{\substack{j=1\\ j\ne i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| =$$
$$= \max_i \frac{1}{|a_{ii}|} \underbrace{\sum_{\substack{j=1\\ j\ne i}}^{n} |a_{ij}|}_{<|a_{ii}|} < \max_i \frac{|a_{ii}|}{|a_{ii}|} = 1$$

Answers

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~~1 Exercise 1~~

b Sheets FundItPowMeth and lab session 2. Since the two biggest eigenvalues are different we know that both have a Jordan block of size 1 or equivalently have each an eigenvector. Now we know that the Jordan matrix is of the form

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & J_s \end{bmatrix}$$

where J_s is again a Jordan matrix containing all the remaining eigenvalues on the diagonal. These are all less in magnitude than λ_1. So as

$$\begin{aligned} x^{(m)} &= A^m x_0 = r_\sigma(A)^m V (J/r_\sigma(A))^m V^{-1} x_0 \\ &\to \lambda_1^m [v_1, v_2] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^m \begin{bmatrix} (V^{-1}x_0)_1 \\ (V^{-1}x_0)_2 \end{bmatrix} \\ &= \lambda_1^m \{(V^{-1}x_0)_1 v_1 + (-1)^m (V^{-1}x_0)_2 v_2\} \end{aligned}$$

Indeed we see that indeed we are ending up in the space spannend by v_1 and v_2 and $V^{-1}x_0$ tells us which vector is approximated at the even steps and which at the odd steps.

Alternative is to write x_0 as a linear combination of (generalized) eigenvectors, which expressed in the above is just $x_0 = V y_0$.

c For linearly converging methods we have eventually $x^{(n+1)} - x \approx \lambda_1 (x^{(n)} - x)$ $\quad \lambda_1$ biggest eigenvalue of iteration matrix

$$\to \quad x^{(n+1)} - x = \lambda_1 (x^{(n+1)} - x + x^{(n)} - x^{(n+1)})$$
$$(1 - \lambda_1)(x^{(n+1)} - x) = \lambda_1 (x^{(n)} - x^{(n+1)})$$
$$x^{(n+1)} - x = \frac{\lambda_1}{1-\lambda_1} (x^{(n)} - x^{(n+1)})$$

if λ_1 close to 1 then

$$\underbrace{|x^{(n+1)} - x|}_{\text{real error}} \gg \underbrace{|x^{(n)} - x^{(n+1)}|}_{\text{difference between subsequent iterates}}$$

Exercise 2

Below the roots the students do not need to compute

```
f1 = x + y - 1; f2 = 0.01 + Log[1 + y - x];
roots = FindRoot[{f1 == 0, f2 == 0}, {x, 0}, {y, 1}]
```

{x → 0.504975, y → 0.495025}

(a) The Jacobian of f

```
f1x = D[f1, x];
f1y = D[f1, y];
f2x = D[f2, x];
f2y = D[f2, y];
Jacf = {{f1x, f1y}, {f2x, f2y}}
MatrixForm[Jacf]
```

$$\left\{\{1, 1\}, \left\{-\frac{1}{1-x+y}, \frac{1}{1-x+y}\right\}\right\}$$

$$\begin{pmatrix} 1 & 1 \\ -\frac{1}{1-x+y} & \frac{1}{1-x+y} \end{pmatrix} \leftarrow$$

(b) 1/2,1/2 is a reasonable guess because both f_1 and f_2 are nearly zero then.

(c) The system which should be used to determine a good A is that where the Jacobian of g is zero in the give approximate fixed point : I+A.Jacf(1/2,1/2)=0

(d)

Below the associated computation that needn't be done by the students.

First we compute A and next we use the earlier found roots to determine the Jacobian of g at the fixed point

```
Jf = Jacf /. {x → 0.5, y → 0.5};
MatrixForm[Jf];
MatrixForm[N[Jf]]
A = -Inverse[Jf];
JacG = Simplify[IdentityMatrix[2] + A.Jacf]
Simplify[JacG /. {x → 0.5, y → 0.5}]
JGp = JacG /. roots
MatrixForm[JGp]
```

$$\begin{pmatrix} 1. & 1. \\ -1. & 1. \end{pmatrix}$$

$$\left\{\left\{0.5 - \frac{0.5}{1-x+y}, -0.5 + \frac{0.5}{1-x+y}\right\}, \left\{-0.5 + \frac{0.5}{1-x+y}, 0.5 - \frac{0.5}{1-x+y}\right\}\right\}$$

{{0., 0.}, {0., 0.}}

{{-0.00502508, 0.00502508}, {0.00502508, -0.00502508}}

Next we compute the eigenvalues and the norm

```
eiv = Eigenvalues[JGp]
Abs[eiv]
Norm[JGp, Infinity]
```

{-0.0100502, 0.}

{0.0100502, 0.}

0.0100502

The infinity norm of the Jacobian of g is less than 1, hence all the eigenvalues are less than 1 according to Th. F.8. Hence, since

(x_n - p)=J_g (x_{n-1}-p)

close to the zero, we will have convergence. Als ~~de eigenwaarden van J_g kleiner dan 1 zijn in abs. waarde dan~~ ~~convergeert~~ dit.

3 a It is just a generalisation of the Taylor expansion error term:

Taylor $f(x) = f(x_0) + (x-x_0)f'(x) + \cdots \frac{(x-x_0)^n}{n!} f^{(n)}(x)$

$$+ \underbrace{(x-x_0)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!}}$$

Error term Taylor exp.
ξ on smallest interval containing $\{x, x_0\}$

→ Interpolation error

$$(x-x_0)(x-x_1) \cdots (x-x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

with ξ on smallest interval containing $\{x_0, x_1 \cdots x_n\}$

In current case

$$x(x-1)(x-2) \frac{f^{(3)}(\xi)}{3!}$$

f is a polynomial of degree 2 → $f^{(3)}(x) \equiv 0$

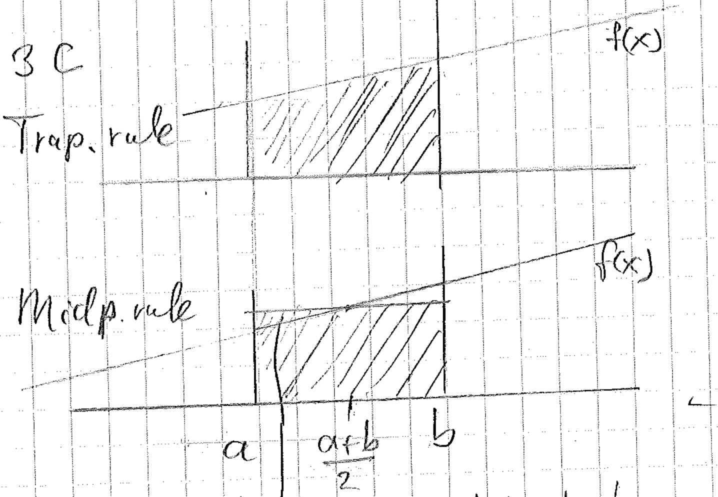
As might be expected a parabola is fixed by three points → Error = 0

b

$$\int_a^b f(x)\,dx = f\left(\frac{a+b}{2}\right)(b-a)$$

Interpolating polynomial: $p_0(x) = f\left(\frac{a+b}{2}\right)$

3 C



what is added here too much is subtracted on the second half of the interval. So midpoint rule also exact for linear functions

3 d

$$
\begin{aligned}
I &= I(h) + c\,h^4 + O(h^5) && \cdot 16\\
I &= I(2h) + c\,(2h)^4 + O(h^5) &&\\
\Rightarrow I &= I(2h) + 16c\,h^4 + O(h^5) && \cdot (-1)\\
\hline
15I &= 16\,I(h) - I(2h) + O(h^5) &&\\
I &= \underbrace{\frac{16\,I(h) - I(2h)}{15}}_{O(h^5)\text{ approximation of } I} + O(h^5) &&
\end{aligned}
$$

4 a Taylor

$$u(x_{i-1}) = u(x_i) - \Delta x\, u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(\xi) + \dots \quad \xi \text{ on } [x_{i-1}, x_i]$$

Rewriting leads to

$$u_x(x_i) = \frac{u(x_i) - u(x_{i-1})}{\Delta x} + \underbrace{\frac{\Delta x}{2} u_{xx}(\xi)}_{O(\Delta x)}$$

Adding extra arguments to u does not influence result.

4. b

$$\frac{d}{dt} u_i(t) = -(u_i(t) - u_{i-1}(t))/\Delta x \qquad i = 2, \dots, m$$

$$\frac{d}{dt} u_1(t) = -(u_1(t) - u_0(t))/\Delta x = -(u_1(t) - \sin^2(t))/\Delta x$$

Initial condition

$$u_i(0) = \sin(\pi x_i)$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} -(u_1 - \sin^2(t))/\Delta x \\ -(u_2 - u_1)/\Delta x \\ \vdots \\ -(u_i(t) - u_{i-1}(t))/\Delta x \\ \vdots \\ -(u_m(t) - u_{m-1}(t))/\Delta x \end{bmatrix} =$$

$$= -\frac{1}{\Delta x}\underbrace{\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix}}_{\downarrow}\begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_m \end{bmatrix} + \underbrace{\frac{1}{\Delta x}\begin{bmatrix} \sin^2(t) \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{\swarrow} =$$

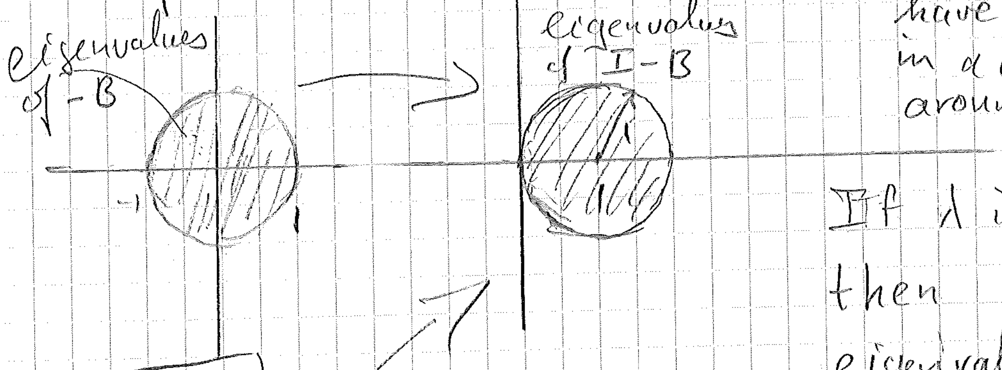
$$= -\frac{1}{\Delta x}(I - B)\vec{u} + \vec{b}(t)$$

$$\Rightarrow \quad B = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

4:c From Formulasheet

$\rho(B) \leq \|B\|_\infty = 1$

B and also $-B$ have all eigenvalues in a circle with radius 1 around the origin



If λ is an eigenvalue of B then $1-\lambda$ is an eigenvalue of $I-B$

$\rightarrow$ all eigenvalues of $-B$ are shifted 1 to the right!

$\boxed{\text{Alternative}}$
All eigenv. of B are zero $\Rightarrow$
All eigenv of $I-B$ are 1

4:d Forward Euler:
$\vec{w}_{n+1} = \vec{w}_n + h\vec{f}(t_n, \vec{w}_n)$

Test equation $y' = \lambda y \rightarrow f(t,y) = \lambda y$

represents eigenvalue of Jacobian of $\vec{f}$

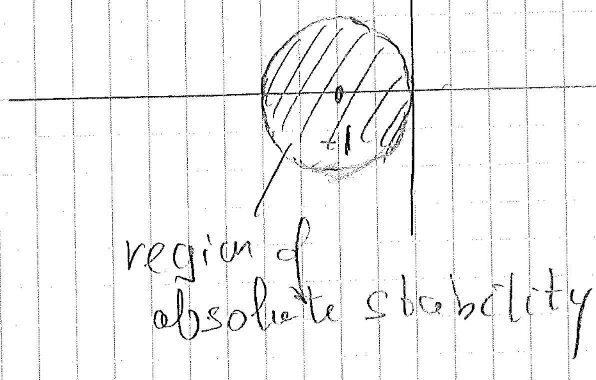
$w_{n+1} = w_n + \Delta t \lambda w_n = (1+\Delta t\lambda) w_n$

$\rightarrow$ Region absolute stability follows from

$|1+z| \leq 1$

$|z-(-1)| \leq 1$

Distance of z to -1 should be less than 1



4.d continued

In our case $\vec{f}(t,\vec{u}) = -\frac{1}{\Delta x}(I-B)\vec{u} + \vec{b}(t)$

This is a linear expression hence Jacobian of $\vec{f}$

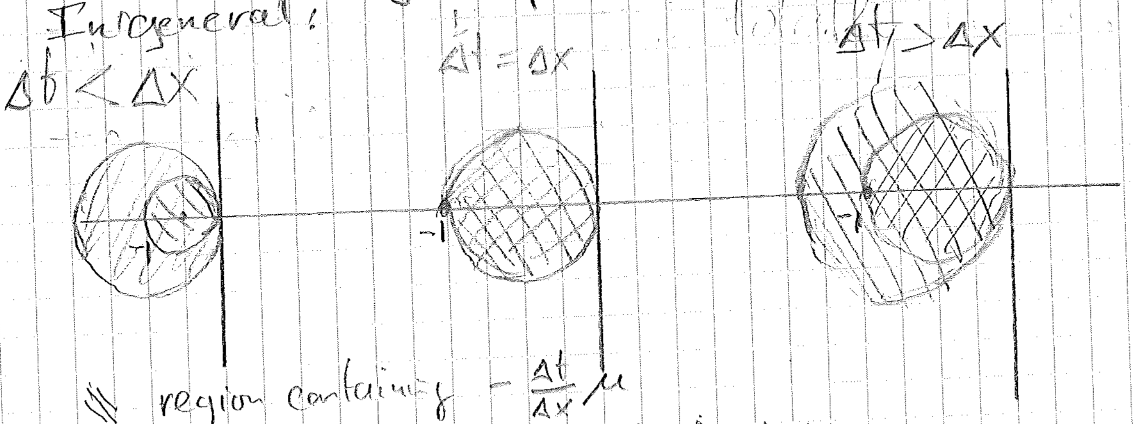
$$J_f = -\frac{1}{\Delta x}(I-B)$$

We know where the eigenvalues of $(I-B)$ are located. If μ eigenv of $I-B$ then $-\frac{\mu}{\Delta x}$ is an eigenvalue of J_f

Now $-\frac{\Delta t}{\Delta x}\mu$ should be in the region of absolute stability

If $\frac{\Delta t}{\Delta x} = 1$ the circle containing all the eigenvalues $-(I-B)$ coincides with the region of absolute stability

In general:



region containing $-\frac{\Delta t}{\Delta x}\mu$

region of absolute stability

So for $\Delta t \leq \Delta x$ we have a stable integration.

Alternative: since all eigen v. of B are zero, the eigenvalues of ~~Δt~~ J_f are all $-\frac{1}{\Delta x}$

to be in region of Abs. stability: $\frac{\Delta t}{\Delta x} \leq 2$

~~Hence~~ There is a reason why this is differing by a factor 2